Closed-Form Stiffness Matrix for the Four-Node Quadrilateral Element with a Fully Populated Material Stiffness

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Abstract: This technical note presents closed-form finite-element stiffness formulations for the four-node quadrilateral element with a fully populated material stiffness, which is required for the nonlinear analysis of reinforced concrete membrane structures. With the material stiffness matrix accounting for anisotropy of the materials and prestrain effects, the developed closed-form element stiffness can be incorporated into a nonlinear finite-element algorithm. Through use of the developed explicit expressions, the examples provided show that the computational effort required to form the stiffness matrix is greatly reduced, compared to either the conventional numerical integration scheme or the elastic-material-stiffness-oriented Griffths' FORTRAN subroutine.

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Introduction

The preference for using the closed-form stiffness matrices in finite-element analysis has long been recognized, while the conventional numerical integration method has been widely employed mainly because of its simplicity. In addition to the danger of matrix rank deficiency due to the use of insufficient integration points, the computational effort in forming the stiffness matrix will be of much importance, particularly in an *h*-adaptive finite-element procedure in which the shape of individual elements are often distorted in adaptively regenerated meshes (Lee and Hobbs 1998). Thus, the gain in computational efficiency by using a closed-form stiffness matrix instead of the conventional numerical integration scheme is of interest for many researchers.

For some displacement type elements, the presence of the rational terms (due to nonconstant Jacobian's determinant in the denominator) in the integrand leads to difficulty in obtaining closed-form (exact integration) expressions for the stiffness matrices. Specifically, for the four-node quadrilateral element, Griffths (1994) proposed a FORTRAN subroutine for calculation of the stiffness matrix for the linear elastic isotropic material.

However, to reflect the nonlinear behavior of reinforced concrete, especially after cracking, the material stiffness must be constructed according to an appropriate set of constitutive relations and the type of stiffness modules employed. A realistic set of formulations were reported in detail by Vecchio (1990), which were based on the modified compression field theory (MCFT) (Vecchio and Collins 1986) and assumed a secant stiffness approach. In doing so, compression softening and hardening, tension stiffening and softening, as well as curvilinear response can be realistically and easily taken into account. The resulting material stiffness matrix will be usually fully populated (3 by 3).

This technical note presents a calculation procedure by which the closed-form stiffness matrix for the four-node quadrilateral element with a fully populated material stiffness can be obtained. The expressions are derived by expanding and simplifying the four terms in the two by two Gauss quadrature, with the help of the computer algebra systems for the algebraic operations. In this aspect, this work is reminiscent of Griffths' FORTRAN subroutine.

Element Stiffness Matrix Calculations

With the material stiffness matrix constructed, one can evaluate the element stiffness matrix using a standard procedure for displacement type of elements. For completeness and to facilitate the derivation of the explicit expression, a brief description of this specific bilinear four-node quadrilateral element will be given first. While the formulations will be derived in closed form, dependent on nodal coordinates and material coefficients only, the concept of isoparametric element is used to describe the procedure of derivation.

For a four-node quadrilateral element shown in Fig. 1, the Jacobian matrix **J** for the isoparametric mapping between global and natural coordinate systems can be expressed as

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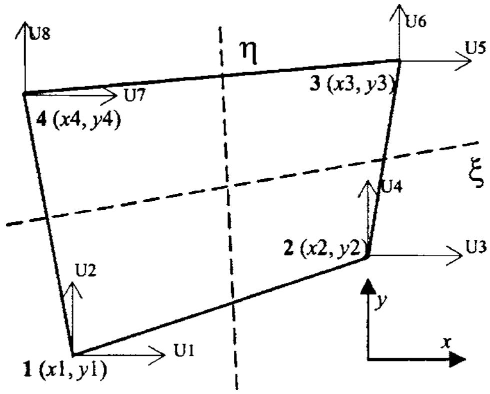

Fig. 1. Schematic four-node quadrilateral element

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \sum_i N_{i,\xi} x_i & \sum_i N_{i,\xi} y_i \\ \sum_i N_{i,\eta} x_i & \sum_i N_{i,\eta} y_i \end{bmatrix} \tag{1}$$

where N_i=shape functions in natural coordinate system and are of the following form

$$N_i = (1 + \xi\xi_i)(1 + \eta\eta_i) \tag{2}$$

The stiffness matrix $\mathbf{K}$ can be written as

$$\mathbf{K} = \int_{V^e} [B]^T [D][B] d(\text{vol}) \tag{3}$$

Here $[B]=[B_1 \quad B_2 \quad B_3 \quad B_4]$, with

$$[B_i] = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \tag{4}$$

$$N_{i,x} = (J_{22} \times N_{i,\xi} - J_{12} \times N_{i,\eta})/\det \mathbf{J} \tag{5}$$

$$N_{i,y} = (J_{11} \times N_{i,\eta} - J_{21} \times N_{i,\xi})/\det \mathbf{J} \tag{6}$$

Now, it is apparent that the difficulty in the exact integration is the presence of a rational term in the integrand of element stiffness matrix, due to nonconstant Jacobian's determinant in the denominator (unlike triangular and rectangular elements where it is twice the area).

The numerical integration element stiffness matrix can be calculated as

$$\mathbf{K} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} ([B]^T [D][B])_{ij} \tag{7}$$

where the subscripts i and j index the integration points.

To reduce the computational effort, some preliminary operations are performed before the detailed calculation procedure, which include grouping of the stiffness matrix's entries, description of coordinate transformations, and analysis of matrix product results. This simplification and decomposition is also necessary to cast the expressions and procedures in a form suitable for documentation.

Due to symmetry, only those entries on and below the main diagonal need to be calculated. Further, it is advantageous to form the matrix product $([B]^T[D][B])$ at each of the Gaussian points while avoiding multiplication by zero since the strain–displacement matrix $[B]$ contains many zero entries. In a partitioned-matrix form, the product can be written as

Table 1. Grouping of Element Stiffness Terms

Group	Freedom character	Stiffness terms	Products included
A	Parallel freedoms at the same node	$k_{11}k_{22}k_{33}k_{44}k_{55}k_{66}k_{77}k_{88}$	$N_{i,x}^2, N_{i,y}^2$
B	Orthogonal freedoms at the same node	$k_{21}k_{43}k_{65}k_{87}$	$N_{i,x}N_{i,y}$
C	Parallel freedoms at the adjacent node	$k_{31}k_{42}k_{53}k_{64}k_{75}k_{86}k_{71}k_{82}$	$N_{i,x}N_{i+1,x}$, $N_{i,y}N_{i+1,y}$
D	Orthogonal freedoms at the adjacent node	$k_{32}k_{41}k_{54}k_{63}k_{76}k_{85}k_{72}k_{81}$	$N_{i,x}N_{i+1,y}$, $N_{i,y}N_{i+1,x}$
E	Parallel freedoms at the opposite node	$k_{51}k_{62}k_{73}k_{84}$	$N_{i,x}N_{i+2,x}$, $N_{i,y}N_{i+2,y}$
F	Orthogonal freedoms at the opposite node	$k_{52}k_{61}k_{74}k_{83}$	$N_{i,x}N_{i+2,y}$, $N_{i,y}N_{i+2,x}$

$$[B]^T [D][B] = \begin{bmatrix} \mathbf{S}_{xx} & \mathbf{S}_{xy} \\ \mathbf{S}_{xy} & \mathbf{S}_{yy} \end{bmatrix} \tag{8}$$

in which

$$\begin{cases} \mathbf{S}_{xx} = D_{11}\mathbf{N}_x^T\mathbf{N}_x + D_{13}(\mathbf{N}_x^T\mathbf{N}_y + \mathbf{N}_y^T\mathbf{N}_x) + D_{33}\mathbf{N}_y^T\mathbf{N}_y \\ \mathbf{S}_{xy} = D_{13}\mathbf{N}_x^T\mathbf{N}_x + D_{12}\mathbf{N}_x^T\mathbf{N}_y + D_{33}\mathbf{N}_y^T\mathbf{N}_x + D_{23}\mathbf{N}_y^T\mathbf{N}_y \\ \mathbf{S}_{yy} = D_{33}\mathbf{N}_x^T\mathbf{N}_x + D_{23}(\mathbf{N}_x^T\mathbf{N}_y + \mathbf{N}_y^T\mathbf{N}_x) + D_{22}N_y^T\mathbf{N}_y \end{cases} \tag{9}$$

where $\mathbf{N}_x$ and $\mathbf{N}_y$=row vectors that contain the partial derivatives of the shape functions with respect to x and y, respectively. Based on the character of freedoms included, the matrix entries can be split into six groups, as originally given by Griffths and shown below

$$\mathbf{K} = \begin{bmatrix} k_{11} \\ k_{21} & k_{22} \\ k_{31} & k_{32} & k_{33} \\ k_{41} & k_{42} & k_{43} & k_{44} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} \\ k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A \\ B & A \\ C & D & A \\ D & C & B & A \\ E & F & C & D & A \\ F & E & D & C & B & A \\ C & D & E & F & C & D & A \\ D & C & F & E & D & C & B & A \end{bmatrix} \tag{10}$$

If one expands the matrices in Eq. (9), it will be found that each group will contain distinct combinations of product $N_{m,x}N_{n,y}$. With detailed characters given in Table 1, it can then be seen that the key calculation in determination of stiffness terms will be in the expressions for these products included. By expanding and simplifying the terms in two by two Gauss quadrature summation, all products $N_{m,x}N_{n,y}$ are of the form

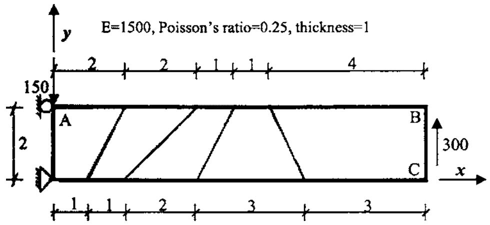

Fig. 2. Problem 1: cantilever beam under end shear

$$\frac{1}{2}\left(\frac{A_2 \cdot s_1 + f_1 \cdot t_1}{3A_2^2 - f_1^2} + \frac{A_2 \cdot s_2 + f_2 \cdot t_2}{3A_2^2 - f_2^2}\right) \tag{11}$$

where

$$A_2 = (x_1 - x_3) \cdot (y_2 - y_4) - (y_1 - y_3) \cdot (x_2 - x_4)$$
$$= \text{twice the area of the element} \tag{12}$$

$$f_1 = (x_1 + x_3) \cdot (y_2 - y_4) - (y_1 + y_3) \cdot (x_2 - x_4) + 2 \cdot (x_2 y_4 - y_2 x_4) \tag{13}$$

$$f_2 = (x_3 - x_1) \cdot (y_2 + y_4) - (y_3 - y_1) \cdot (x_2 + x_4) + 2(x_1 y_3 - y_1 x_3) \tag{14}$$

All other coefficients in Eq. (11) will be formulated only for the defined *parent* product within each group and remaining products can then be calculated through two nodal coordinate transformations as described below

$$\text{T1:} \quad (x_1,y_1) \Rightarrow (x_2,y_2) \Rightarrow (x_3,y_3) \Rightarrow (x_4,y_4) \Rightarrow (x_1,y_1)$$

$$\text{T3:} \quad (x_i,y_i) \Rightarrow (y_i,x_i)$$

Note that transformation T1 will affect the terms f in Eq. (11), in addition to s and t. All detailed calculations are described below, in terms of groups defined in Table 1.

Group A with parent $N_{1,x}^2$

$$s_1 = 2(y_4 - y_2)^2, \quad s_2 = (y_2 - y_3)^2 + (y_3 - y_4)^2 + (y_4 - y_2)^2 \tag{15}$$

$$t_1 = -s_1/2, \quad t_2 = (y_2 - y_3)^2 - (y_3 - y_4)^2 \tag{16}$$

$N_{i+1,x}^2$ is then computed from $N_{i,x}^2$ through a T1 transformation while the T3 transformation is used when computing $N_{i,y}^2$ from $N_{i,x}^2$.

Group B with parent $N_{1,x}N_{1,y}$

$$s_1 = 2(y_4 - y_2)(x_2 - x_4)$$

$$s_2 = x_2(y_4 - 2y_2 + y_3) + x_3(y_2 - 2y_3 + y_4) + x_4(y_2 - 2y_4 + y_3) \tag{17}$$

$$t_1 = -s_1/2, \quad t_2 = x_2(y_3 - y_2) + x_3(y_2 - y_4) + x_4(y_4 - y_3) \tag{18}$$

$N_{i+1,x}N_{i+1,y}$ is then computed from $N_{i,x}N_{i,y}$ through a T1 transformation.

Group C with parent $N_{1,x}N_{2,x}$

$$s_1 = (y_2 - y_4)(2y_1 - y_3 - y_2), \quad s_2 = (y_3 - y_1)(2y_4 - y_3 - y_2) \tag{19}$$

Table 2. Efficiency Test in Problem 1

Scheme	CPU (s)	Speed ratio
Gauss quadrature	0.5470	1.0
Griffths subroutine	0.2190	2.5
Proposed formulation	0.1870	2.9

$$t_1 = (y_2 - y_4)(y_2 - y_1), \quad t_2 = (y_3 - y_1)(y_3 - y_4) \tag{20}$$

$N_{i+1,x}N_{i+2,x}$ is then computed from $N_{i,x}N_{i+1,x}$ through a T1 transformation while the T3 transformation is used when computing $N_{i,y}N_{i+1,y}$ from $N_{i,x}N_{i+1,x}$.

Group D with parent $N_{1,x}N_{2,y}$

$$s_1 = (y_2 - y_4)(x_2 - 2x_1 + x_3), \quad s_2 = (x_3 - x_1)(y_2 - 2y_4 + y_3) \tag{21}$$

$$t_1 = (y_2 - y_4)(x_1 - x_2), \quad t_2 = (x_3 - x_1)(y_4 - y_3) \tag{22}$$

$N_{i+1,x}N_{i+2,y}$ is then computed from $N_{i,x}N_{i+1,y}$ through a T1 transformation while the T3 transformation is used when computing $N_{i,y}N_{i+1,x}$ from $N_{i,x}N_{i+1,y}$.

Group E with parent $N_{1,x}N_{3,x}$

$$s_1 = -(y_2 - y_4)^2$$

$$s_2 = (y_3 + y_1)(y_4 + y_2) - 2(y_2 - y_4)^2 - 2(y_1 y_3 + y_2 y_4) \tag{23}$$

$$t_1 = 0, \quad t_2 = (y_2 - y_4)(y_1 - y_2 + y_3 - y_4) \tag{24}$$

$N_{i+1,x}N_{i+3,x}$ is then computed from $N_{i,x}N_{i+2,x}$ through a T1 transformation while the T3 transformation is used when computing $N_{i,y}N_{i+2,y}$ from $N_{i,x}N_{i+2,x}$.

Group F with parent $N_{1,x}N_{3,y}$

$$s_1 = (y_2 - y_4)(x_2 - x_4)$$

$$s_2 = (x_4 - x_2)(y_4 - y_2) + (x_2 - x_1)(y_2 - y_3) + (x_4 - x_1)(y_4 - y_3) \tag{25}$$

$$t_1 = 0, \quad t_2 = (x_2 - x_1)(y_2 - y_3) + (x_4 - x_1)(y_3 - y_4) \tag{26}$$

$N_{i+1,x}N_{i+3,y}$ is then computed from $N_{i,x}N_{i+2,y}$ through a T1 transformation while the T3 transformation is used when computing $N_{i,y}N_{i+2,x}$ from $N_{i,x}N_{i+2,y}$.

Having calculated these preceding products $N_{m,x}N_{n,y}$, one can readily evaluate the stiffness matrix entries by using Eq. (9). For example

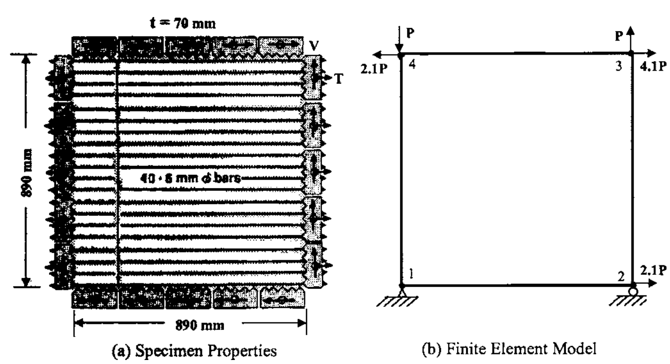

Fig. 3. Problem 2: panel specimen PB21

194530.0	-60625.8	-14140.0	88135.8	-13370.0	-60555.8	-167020.0	33045.8
60625.8	104475.0	61010.8	-104930.0	-60555.8	45640.0	60170.8	-45185.0
14140.0	61010.8	530110.0	-424100.8	-167020.0	60170.8	-348950.0	302919.2
88135.8	-104930.0	-424100.8	495985.0	33045.8	-45185.0	302919.2	-345870.0
13370.0	-60555.8	-167020.0	33045.8	194530.0	-60625.8	-14140.0	88135.8
60555.8	45640.0	60170.8	-45185.0	-60625.8	104475.0	61010.8	-104930.0
167020.0	60170.8	-348950.0	302919.2	-14140.0	61010.8	530110.0	-424100.8
33045.8	-45185.0	302919.2	345870.0	88135.8	-104930.0	-424100.8	495985.0

Fig. 4. Element stiffness matrix output in Problem 2 (in N/mm)

$$k_{47} = D_{13}\mathbf{N}_{2,x}\mathbf{N}_{4,x} + D_{12}\mathbf{N}_{4,x}\mathbf{N}_{2,y} + D_{33}\mathbf{N}_{4,y}^{T}\mathbf{N}_{2,x} + D_{23}\mathbf{N}_{2,y}^{T}\mathbf{N}_{4,y} \quad (27)$$

Note that the degrees of freedoms are numbered in the manner as shown in Fig. 1.

Efficiency Tests

To assess the computational performance of the explicit formulations presented, this section will present two problems for comparison and discussion.

1. Problem 1: Linear elastic element. Consider a cantilever beam under shear force, as shown in Fig. 2. This problem is analyzed by five distorted quadrilaterals which are linear elastic elements. The computational efficiency of the developed formulation is compared to both the conventional Gaussian quadrature scheme and Griffths' procedure, in terms of CPU time.

   To avoid the computational overhead associated with other processes, such as assembly, only the element stiffness formation is timed. In addition, the stiffness matrix is evaluated repeatedly up to 5,000 times for all five different quadrilaterals. Table 2 shows the CPU time logged on a scalar machine and the speedup ratio calculated. The results obtained indicate that the computational effort required to form the stiffness matrix is greatly reduced by using the developed procedure, compared to either the conventional numerical integration scheme or the elastic-material-stiffness-oriented Griffths' FORTRAN subroutine.
2. Problem 2: Nonlinear anisotropic element. This problem examines the panel specimen tested by Bhide and Collins and analyzed by Vecchio. As given in Fig. 3, the finite element used is rectangular in shape and with nonlinear anisotropy in material. The material properties and specified uniform stress conditions follow those given in the reference (Vecchio 1990).

   The task is to compare the CPU time required to form the element stiffness matrices using different schemes. Among these schemes are the rectangular exact integration scheme, the Gaussian quadrature scheme, and the proposed closed-form formulation. All schemes produce the same element stiffness matrix in Fig. 4 (round to the first digit after the decimal place).

The CPU time logged, given in Table 3, shows that the developed scheme is slower than the rectangular-element-oriented exact integration scheme but again much faster than the numerical integration scheme. Note, however, that the rectangular element is not suitable in many applications; for example, in nonuniform mesh topologies, or where nonlinear geometry effects result in mesh regeneration. For such situations, the proposed quadrilateral element can effectively be substituted. Also, though the MCFT relations are employed in the construction of material stiffness, the element stiffness formulations presented are such that any realistic set of constitutive relations can be easily implemented.

Table 3. Efficiency Test in Problem 2

Number of evaluation	CPU time (s) Gauss quadrature	Exact integration	Proposed formulation	Speed ratio[a]
5,000	0.1100	0.0083	0.0468	1:13.3:2.4
10,000	0.2180	0.0160	0.0780	1:13.7:2.8
100,000	2.1870	0.1460	0.7810	1:14.9:2.8

[a]Calculated in terms of "1:m:n" with Gauss scheme being the reference.

Conclusions

In this work, an explicit procedure for the closed-form element stiffness matrix is derived for the four-node quadrilateral element with a fully populated material stiffness, which realistically models the nonlinear behavior of reinforced concrete membrane structures. The algebraic expressions are based on expanding and simplifying the terms in the summation of the numerical integration. While the formulation elegance and computational effort are argued to be of primary importance, the developed scheme has a compact expression and good computational efficiency. It is believed that the formulations are advantageous in situations where the closed-form rectangular element cannot be used.

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